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ELASTIC MODULI OF TWO-DIMENSIONAL COMPOSITES WITH SLIDING INCLUSIONS— A COMPARISON OF EFFECTIVE MEDIUM THEORIES

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Abstract—We study the effective elastic moduli of two-dimensional (2D) composite materials containing sliding circular inclusions distributed randomly in the matrix. To simulate sliding we introduce a sliding parameter, which in two limiting cases gives perfect bonding and pure sliding boundary conditions. We evaluate elastic moduli using four effective medium theories; the self-consistent method, the differential scheme, the Mori-Tanaka method and the generalized self-consistent method. In this paper we focus on two aspects: one is the study of the effect of interface on the elastic constants of composites and the other is a comparison of the results from effective medium theories for the cases of both sliding and perfect bonding. In the discussion we use the recently-stated Cherkaev–Lurie–Milton theorem, which gives general relations between the effective elastic constants of 2D composites. We also compare the results from the effective medium theories with those from numerical simulations.

1. INTRODUCTION

In this paper we evaluate the effective elastic moduli of two-dimensional (2D) composite materials which are reinforced with inclusions dispersed randomly in the matrix. We assume that the inclusions are circular in shape and we allow sliding at the matrix-inclusion interfaces.

The exact theoretical evaluation of the effective elastic properties of composites is in general very difficult, because it requires a precise knowledge of the stress and strain fields everywhere in the composite. Therefore, many micromechanics-based models have been introduced to make such a problem mathematically tractable. However, the simplifying assumptions used in these models may give rise to considerable differences in the predictions of such methods as shown recently by Christensen (1990) and Christensen *et al.* (1992), for example. The methods which are widely used include: the self-consistent method (Bruggeman, 1935; Budiansky, 1965; Hill, 1965), the differential scheme (Bruggeman, 1935; Roscoe, 1952), the Mori-Tanaka method (Benveniste, 1987; Mori and Wakashima, 1990), the generalized self-consistent method (Mackenzie, 1950; Christensen and Lo, 1979), and the composite cylinder assemblage model (Hashin and Rosen, 1964). Since in this paper we are interested in the explicit expressions and the composite cylinder model does not give a single result for the transverse shear modulus but the bounds, we do not include this model in the mainstream discussion.

In this paper we predict the elastic properties of composites with sliding interfaces by using the four above mentioned effective medium theories. We focus on two aspects: the study of the effect of interface on the elastic constants and the comparison of results from these methods. We model sliding by using a sliding parameter which relates tangential tractions to the jump in tangential displacements, while the continuity of normal displacements and tractions is maintained. This boundary condition gives perfect bonding and pure sliding conditions in limiting cases. Our paper is related to recent studies of Christensen (1990) and Zimmerman (1991), which were done for the perfect bonding case, and is an extension of the work of Jasiuk *et al.* (1992a), who studied the effective elastic constants of composites with rigid sliding inclusions using the self-consistent method and the differential scheme. Here, we consider a more general case of elastic inclusions and use four effective medium theories. In particular, we are interested in several limiting cases involving holes, rigid inclusions, constituents with equal shear moduli, or equal Poisson's ratio, and a material with sliding inclusions but same elastic constants. These limiting cases exhibit interesting physical behavior, and because of a smaller parameter space enable us to compare the above methods more easily. The rigid case is of particular interest because it yields the largest differences between the methods and shows the greatest effect of sliding.

We limit our analysis to the 2D geometry so we can relate our results to a recent Cherkaev–Lurie–Milton theorem (Cherkaev *et al.*, 1992) or CLM theorem, which holds only in two dimensions. This theorem is very powerful because it gives the relation between the elastic constants which is independent of the details of microstructure. The CLM theorem also gives a general result for stress fields which agrees with the results of Dundurs (1967, 1970) and Michell (1899), as discussed by Thorpe and Jasiuk (1992). We show that all effective medium theories studied in this paper do satisfy the CLM theorem for both perfect bonding and sliding. The original proof of the CLM theorem, given in Cherkaev *et al.* (1992), was based on the assumption that all interfaces are perfectly bonded. However, recently Moran and Gosz (1992) showed that the CLM theorem also holds for the sliding case. We include a different version of their proof in the Appendix B, for completeness. This proves an earlier conjecture of Thorpe and Jasiuk (1992) about the validity of the CLM theorem for more general boundary conditions.

Finally, we compare the results of effective medium theories with those from numerical simulations (Chen *et al.*, 1993; Day *et al.*, 1992; Snyder *et al.*, 1992).

2. TWO-DIMENSIONAL ELASTICITY

The stress–strain relations for a linear elastic and isotropic material in three-dimensions (3D) are

$$\varepsilon_{ij} = \frac{1}{E'}[(1+v')\sigma_{ij} - v'\sigma_{kk}\delta_{ij}] \quad i,j,k = 1,2,3, \tag{1}$$

where ε_{ij} and σ_{ij} are the strain and stress tensors, respectively, and E' and v' are Young's modulus and Poisson's ratio, respectively. Here, we follow the same notation as in Thorpe and Jasiuk (1992) and we use primes to denote the quantities in 3D, so we can use the unprimed quantities in 2D.

In 2D or plane elasticity, the constitutive equations can be expressed as

$$\varepsilon_{ij} = \frac{1}{E}[(1+v)\sigma_{ij} - v\sigma_{kk}\delta_{ij}] \quad i,j,k = 1,2, \tag{2}$$

where E is the 2D (area or planar) Young's modulus and v is the 2D (area) Poisson's ratio. In 2D, the upper bound on v is 1, as opposed to 1/2 for v' in 3D.

The area bulk modulus K and the shear modulus μ are defined in terms of 2D constants as (Sen and Thorpe, 1985; Thorpe and Jasiuk, 1992)

$$K = \frac{E}{2(1-v)}, \tag{3}$$

$$\mu = \frac{E}{2(1+v)}. \tag{4}$$

Two other useful relations are

$$\frac{4}{E} = \frac{1}{K} + \frac{1}{\mu}, \tag{5}$$

$$v = \frac{K-\mu}{K+\mu}. \tag{6}$$

If we distinguish between the plane stress and plane strain cases, the 2D elastic constants are related to 3D elastic constants for plane strain as

$$E = \frac{E'}{1-\nu'^2}, \quad \nu = \frac{\nu'}{1-\nu'}, \quad K = K' + \frac{\mu'}{3} \tag{7}$$

and for plane stress as

$$E = E', \quad \nu = \nu', \quad K = \frac{9K'\mu'}{3K'+4\mu'} \tag{8}$$

and for both cases

$$\mu = \mu', \tag{9}$$

where K' and μ' are the 3D bulk and shear moduli, respectively.

In the remaining part of this paper we use the 2D elastic constants.

3. PROBLEM STATEMENT

We consider a plane elasticity problem and study the effective elastic moduli of 2D composite materials reinforced with circular inclusions distributed uniformly at random (i.e. with spatially homogeneous statistics) in the matrix. An example of a 2D composite is a thin sheet with disks (plane stress) or a transverse plane of a unidirectional composite (plane strain), for example. Both the inclusions and the matrix are homogeneous, linear elastic and isotropic. The interfaces between the inclusions and the matrix allow sliding (slip).

The sliding boundary conditions between the matrix and inclusions involve continuity of tractions and of normal displacements, and discontinuity in tangential displacements across the interface. If we employ a polar coordinate system (r, θ) with the origin at the center of a circular inclusion of radius a, then the sliding boundary conditions at the inclusion-matrix interface are

$$\sigma^{\mathrm{m}}_{rr}(a,\theta) = \sigma^{\mathrm{f}}_{rr}(a,\theta), \tag{10}$$

$$u^{\mathrm{m}}_{r}(a,\theta) = u^{\mathrm{f}}_{r}(a,\theta), \tag{11}$$

$$\sigma^{\mathrm{m}}_{r\theta}(a,\theta) = \sigma^{\mathrm{f}}_{r\theta}(a,\theta) = k[u^{\mathrm{m}}_{\theta}(a,\theta) - u^{\mathrm{f}}_{\theta}(a,\theta)], \tag{12}$$

where the superscripts f and m denote the inclusion (fiber) and the matrix, respectively. The sliding parameter k is a measure of the degree of sliding at the interface. The pure sliding condition is reached when k is zero and perfect bonding case when k goes to infinity. These "spring type" boundary conditions have been used by Lene and Leguillon (1982), Benveniste (1985), Kouris and Mura (1989), Jasiuk *et al.* (1992a), and others.

This interfacial model can be generalized to also include a condition that normal tractions are proportional to the jumps in normal displacements (Jones and Whittier, 1967; Aboudi, 1987; Steif and Hoysan, 1987; Achenbach and Zhu, 1989, 1990; Jasiuk and Tong, 1989; Hashin, 1990, 1991, 1992; Zhu and Achenbach, 1991; Thorpe and Jasiuk, 1992). Then, eqns (10)–(11) are replaced by

$$\sigma^{\mathrm{m}}_{rr}(a,\theta) = \sigma^{\mathrm{f}}_{rr}(a,\theta) = m[u^{\mathrm{m}}_{r}(a,\theta) - u^{\mathrm{f}}_{r}(a,\theta)]. \tag{13}$$

However, this condition needs to be used with caution because of possible overlapping of materials. We choose an earlier model for simplicity, to reduce the parameter space, and to address the problem of sliding in particular.

We may add that the interfacial model considered here (10)–(12) is a special case of boundary conditions (12)–(13) with $m \to \infty$.

3.1. *Dilute result*

When the concentration of inclusions is very small (dilute) and there is no interaction between the inclusions, the effective elastic moduli can be predicted exactly. The dilute result can be derived by using the solution of an isolated inclusion and the equivalence of elastic strain energies (Christensen, 1979).

When a remote stress field σ_{ij}^0 is applied to the domain D containing a single sliding inclusion Ω, the elastic strain energy is expressed as (Jasiuk *et al.*, 1992a)

$$W = W^0 + \frac{1}{2}\int_{|\Omega|} (\sigma_{ij}^0 n_j u_i - \sigma_{ij} n_j u_i^0)\,\mathrm{d}S + \frac{1}{2}\int_{|\Omega|} \sigma_{ij}^0 n_j [u_i]\,\mathrm{d}S, \tag{14}$$

where $W^0 = \frac{1}{2}\int_D \sigma_{ij}^0 \varepsilon_{ij}^0\,\mathrm{d}V$ and $|\Omega|$ is the surface of the inclusion. The superscript 0 denotes quantities due to the applied loads in the absence of inclusion and the subscripts i, j denote the general coordinates. The quantities σ_{ij} and u_i imply the total stresses and displacements which include σ_{ij}^0, u_i^0 and the disturbance due to the presence of inclusions. n_j represents a unit vector which is normal to the inclusion-matrix interface and $[u_j] = u_j^{\mathrm{m}} - u_j^{\mathrm{f}}$ is the jump in displacements at the interface.

The elastic strain energy stored in the equivalent homogeneous medium is

$$\frac{W}{D} = \frac{1}{2}\sigma_{ij}^0 S_{ijkl} \sigma_{kl}^0, \tag{15}$$

where S_{ijkl} is the effective compliance.

Using the boundary conditions (10)–(12) and equating the elastic strain energies in eqns (14) and (15), the effective area bulk modulus K and the effective shear modulus μ have been found to be (Thorpe and Jasiuk, 1992),

$$\frac{1}{K} = \frac{1}{K^{\mathrm{m}}} + c\left(\frac{1}{K^{\mathrm{f}}} - \frac{1}{K^{\mathrm{m}}}\right)\left[\frac{\dfrac{1}{K^{\mathrm{m}}} + \dfrac{1}{\mu^{\mathrm{m}}}}{\dfrac{1}{K^{\mathrm{f}}} + \dfrac{1}{\mu^{\mathrm{m}}}}\right], \tag{16}$$

$$\frac{1}{\mu} = \frac{1}{\mu^{\mathrm{m}}} + 2c\left(\frac{1}{\mu^{\mathrm{m}}} + \frac{1}{K^{\mathrm{m}}}\right)\left[\frac{\left(\dfrac{2}{\mu^{\mathrm{f}}} - \dfrac{1}{\mu^{\mathrm{m}}} + \dfrac{1}{K^{\mathrm{f}}}\right) + \tilde{k}\left(\dfrac{1}{\mu^{\mathrm{f}}} - \dfrac{1}{\mu^{\mathrm{m}}}\right)\left(\dfrac{1}{\mu^{\mathrm{m}}} + \dfrac{1}{\mu^{\mathrm{f}}} + \dfrac{2}{K^{\mathrm{f}}}\right)}{\left(\dfrac{2}{\mu^{\mathrm{m}}} + \dfrac{2}{\mu^{\mathrm{f}}} + \dfrac{3}{K^{\mathrm{m}}} + \dfrac{1}{K^{\mathrm{f}}}\right) + \tilde{k}\left(\dfrac{1}{\mu^{\mathrm{m}}} + \dfrac{1}{\mu^{\mathrm{f}}} + \dfrac{2}{K^{\mathrm{m}}}\right)\left(\dfrac{1}{\mu^{\mathrm{m}}} + \dfrac{1}{\mu^{\mathrm{f}}} + \dfrac{2}{K^{\mathrm{f}}}\right)}\right], \tag{17}$$

where $\tilde{k} = (ka)/2$ and c is the volume (area) fraction of inclusions. From these solutions we can obtain two limiting cases, perfect bonding ($\tilde{k} \to \infty$) and pure sliding ($\tilde{k} \to 0$).

3.2. *Self-consistent method*

The self-consistent method (Bruggeman, 1935; Budiansky, 1965; Hill, 1965) assumes that a typical inclusion is embedded in a homogeneous material having the effective elastic constants K and μ.

In order to calculate the effective shear modulus, we apply a remote external shear stress σ_{xy}^0. Then, the volume (area) averages of the total strain ε_{xy} and stress σ_{xy} are given as

$$\langle \varepsilon_{xy} \rangle = \frac{1}{V}\int_{D-\Omega} \varepsilon_{xy}\,\mathrm{d}V + \frac{1}{V}\int_{\Omega} \varepsilon_{xy}\,\mathrm{d}V + \frac{1}{2V}\int_{|\Omega|} ([u_x]n_y + [u_y]n_x)\,\mathrm{d}S, \tag{18}$$

$$\langle \sigma_{xy} \rangle = \frac{1}{V}\int_{D-\Omega} \sigma_{xy}\,\mathrm{d}V + \frac{1}{V}\int_{\Omega} \sigma_{xy}\,\mathrm{d}V = \sigma^0_{xy}, \tag{19}$$

where the subscripts x, y denote rectangular coordinates. The last integral in (18) accounts for sliding and can be calculated by using the sliding boundary conditions (12). After evaluating the integrals in eqns (18)–(19), the average strain and stress can be written as

$$\langle \varepsilon_{xy} \rangle = c\left(\frac{\langle \varepsilon_{xy} \rangle}{g^c}\right)\mu + (1-c)\varepsilon^{\mathrm{m}} + c\left(\frac{\langle \varepsilon_{xy} \rangle}{g^c}\right)\mu\mu^{\mathrm{f}}\left[\frac{\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}}{3+2\tilde{k}\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}\right], \tag{20}$$

$$\langle \sigma_{xy} \rangle = 2c\left(\frac{\langle \varepsilon_{xy} \rangle}{g^c}\right)\mu^{\mathrm{f}}\mu + (1-c)\sigma^{\mathrm{m}}, \tag{21}$$

where g^c is defined as

$$\frac{1}{g^c} = \left[\frac{\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}}{3+2\tilde{k}\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}\right]\left[\frac{\left(\dfrac{\mu}{\mu^{\mathrm{f}}}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{K}\right)}{\left(\dfrac{2}{\mu}+\dfrac{2}{\mu^{\mathrm{f}}}+\dfrac{3}{K}+\dfrac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}\right] \tag{22}$$

and σ^{m}, ε^{m} are the average shear stress and strain in the original matrix with the shear modulus μ^{m}. By using $\sigma^{\mathrm{m}} = 2\mu^{\mathrm{m}}\varepsilon^{\mathrm{m}}$ and eliminating σ^{m} from eqns (20)–(22) the effective shear modulus μ is found to be

$$\frac{1}{\mu} = \frac{1}{\mu^{\mathrm{m}}} + c\left(\frac{1}{\mu}+\frac{1}{K}\right)\left[\frac{\left(\dfrac{1}{\mu}-\dfrac{3}{\mu^{\mathrm{m}}}+\dfrac{4}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)+2\tilde{k}\left(\dfrac{1}{\mu^{\mathrm{f}}}-\dfrac{1}{\mu^{\mathrm{m}}}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}{\left(\dfrac{2}{\mu}+\dfrac{2}{\mu^{\mathrm{f}}}+\dfrac{3}{K}+\dfrac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}\right]. \tag{23}$$

The area bulk modulus can be obtained in a similar way if the external loads $\sigma^0_{xx} = \sigma^0_{yy}$ are applied. The result is

$$\frac{1}{K} = \frac{1}{K^{\mathrm{m}}} + c\left(\frac{1}{K^{\mathrm{f}}}-\frac{1}{K^{\mathrm{m}}}\right)\left[\frac{\dfrac{1}{K}+\dfrac{1}{\mu}}{\dfrac{1}{K^{\mathrm{f}}}+\dfrac{1}{\mu}}\right]. \tag{24}$$

Expressions for the effective shear and bulk modulus are highly coupled and are solved numerically. In the brief presentation here we followed Mura (1987) but we modified the expressions to account for sliding. Related results from the self-consistent method in 2D are reported in Ostoja-Starzewski *et al.* (1993).

The self-consistent method was used earlier for the sliding case by Ghahremani (1980) to predict the elastic moduli of polycrystals with spherical grains having freely sliding boundaries and by Jasiuk *et al.* (1992a) who studied a composite with rigid sliding inclusions.

3.3. *Differential scheme*

The differential scheme was developed by Bruggeman (1935) and Roscoe (1952) and employed by McLaughlin (1977), Norris (1985), Zimmerman (1991), and others.

Let us consider an effective homogeneous medium whose moduli are given by dilute results (16)–(17). The differential scheme suggests a sequential procedure such that a small amount of area fraction of inclusions δc is added to the effective medium at each step. As the area fraction of inclusions increases, the effective moduli change by the increments of δK and $\delta\mu$. The differential equations are set up from the dilute results by taking the limit $\delta c \to 0$ leading to

$$\frac{\mathrm{d}K}{K} = \frac{\mathrm{d}c}{1-c}\left(1-\frac{K}{K^{\mathrm{f}}}\right)\left[\frac{\dfrac{1}{K}+\dfrac{1}{\mu}}{\dfrac{1}{K^{\mathrm{f}}}+\dfrac{1}{\mu}}\right], \tag{25}$$

$$\frac{\mathrm{d}\mu}{\mu} = \frac{2\,\mathrm{d}c}{1-c}\left(1+\frac{\mu}{K}\right)\left[\frac{\left(\dfrac{1}{\mu}-\dfrac{2}{\mu^{\mathrm{f}}}-\dfrac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\dfrac{1}{\mu}-\dfrac{1}{\mu^{\mathrm{f}}}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}{\left(\dfrac{2}{\mu}+\dfrac{2}{\mu^{\mathrm{f}}}+\dfrac{3}{K}+\dfrac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K}\right)\left(\dfrac{1}{\mu}+\dfrac{1}{\mu^{\mathrm{f}}}+\dfrac{2}{K^{\mathrm{f}}}\right)}\right]. \tag{26}$$

Equations (25) and (26) are two highly coupled nonlinear differential equations. The solution can be obtained by using the conditions $\mu = \mu^{\mathrm{m}}$ and $K = K^{\mathrm{m}}$ at $c = 0$, as illustrated by Jasiuk *et al.* (1992a), for example.

3.4. *Mori–Tanaka method*

This average field method was introduced by Mori and Tanaka (1973) and has been used by a number of researchers to predict the effective properties. There are two formulations of the Mori–Tanaka method for the perfect bonding case, which have different derivations but yield identical results. One is due to Benveniste (1987) and involves the "strain" or "stress concentration factor" concept. The second approach involves an equivalent inclusion method (Eshelby, 1957), and was recently reformulated by Mori and Wakashima (1990) and referred to as a successive iteration method. Both formulations can be extended to the sliding case. In this paper we follow the approach of Mori and Wakashima (1990), which was generalized by Shibata *et al.* (1990) to solve the case of a material with freely sliding inclusions having same elastic constants as the matrix.

If a traction $t_i = \sigma_{ij}^0 n_j$ is applied at infinity, where σ_{ij}^0 is constant, the average stress field in a single inclusion Ω is given as

$$\sigma_{ij}^0 + \langle\tilde{\sigma}_{ij}\rangle_\Omega = C_{ijkl}^{\mathrm{f}}(\varepsilon_{kl}^0 + \langle\tilde{\varepsilon}_{kl}\rangle_\Omega) = C_{ijkl}^{\mathrm{m}}(\varepsilon_{kl}^0 + \langle\tilde{\varepsilon}_{kl}\rangle_\Omega - \langle\varepsilon_{kl}^{0*}\rangle_\Omega), \tag{27}$$

where ε_{ij}^{0*} is the fictitious eigenstrain in the isolated inclusion, $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{ij}$ are stress and strain disturbances in an isolated inclusion, and $< >$ implies the volume (area) average. This is the equivalent inclusion method (Eshelby, 1957; Mura, 1987) which is usually applied to the perfect bonding case, but is also valid for the sliding case when the volume averages are used. The equivalent eigenstrain in the sliding inclusion can be calculated by equating the elastic strain energies of the inclusion, with the eigenstrain ε_{ij}^{0*}, having the same elastic constants as the matrix and of the inclusion with different elastic constants and subjected to a remote load σ_{ij}^0

$$\frac{1}{2}\int_{\Omega}\sigma_{ij}^{0}\varepsilon_{ij}^{0}\,\mathrm{d}V+\frac{1}{2}\int_{\Omega}\sigma_{ij}^{0}\varepsilon_{ij}^{0*}\,\mathrm{d}V=\frac{1}{2}\int_{\Omega}\sigma_{ij}^{0}\varepsilon_{ij}^{0}\,\mathrm{d}V+\frac{1}{2}\int_{\Omega}(\sigma_{ij}^{0}\varepsilon_{ij}-\sigma_{ij}\varepsilon_{ij}^{0})\,\mathrm{d}V+\frac{1}{2}\int_{|\Omega|}\sigma_{ij}n_{j}[u_{i}]\,\mathrm{d}S. \tag{28}$$

This can be rewritten as

$$\frac{\sigma_{ij}^{0}}{\Omega}\int_{\Omega}\varepsilon_{ij}^{0*}\,\mathrm{d}V=\sigma_{ij}^{0}\langle\varepsilon_{ij}^{0*}\rangle_{\Omega}=\frac{1}{\Omega}\int_{\Omega}(\sigma_{ij}^{0}\varepsilon_{ij}-\sigma_{ij}\varepsilon_{ij}^{0})\,\mathrm{d}V+\frac{1}{\Omega}\int_{|\Omega|}\sigma_{ij}n_{j}[u_{i}]\,\mathrm{d}S, \tag{29}$$

where $\langle\varepsilon_{ij}^{0*}\rangle_{\Omega}$ is the average equivalent eigenstrain in the isolated inclusion.

The effective shear modulus is obtained when a remote shear stress σ_{xy}^{0} is applied to the medium. Then, by definition,

$$\frac{\sigma_{xy}^{0}}{2\mu}=\frac{\sigma_{xy}^{0}}{2\mu^{\mathrm{m}}}+c\langle\varepsilon_{xy}^{*}\rangle_{\Omega}, \tag{30}$$

where $\langle\varepsilon_{xy}^{*}\rangle_{\Omega}$ is the average equivalent eigenstrain in the inclusion which includes the interaction with other inclusions and can be evaluated from the isolated inclusion solution as

$$\langle\varepsilon_{xy}^{*}\rangle_{\Omega}=\frac{\beta}{1+c\alpha}\sigma_{xy}^{0}, \tag{31}$$

where α and β are defined as

$$\langle\tilde{\sigma}_{xy}\rangle_{\Omega}=\alpha\sigma_{xy}^{0}, \tag{32}$$

$$\langle\varepsilon_{xy}^{0*}\rangle_{\Omega}=\beta\sigma_{xy}^{0}, \tag{33}$$

where

$$\alpha=-\frac{A}{2},$$

$$\beta=\frac{A}{2}\left(\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{K^{\mathrm{m}}}\right), \tag{34}$$

with

$$A=\frac{\left(\frac{2}{\mu^{\mathrm{f}}}-\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\frac{1}{\mu^{\mathrm{f}}}-\frac{1}{\mu^{\mathrm{m}}}\right)\left(\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{\mu^{\mathrm{f}}}+\frac{2}{K^{\mathrm{f}}}\right)}{\left(\frac{2}{\mu^{\mathrm{m}}}+\frac{2}{\mu^{\mathrm{f}}}+\frac{3}{K^{\mathrm{m}}}+\frac{1}{K^{\mathrm{f}}}\right)+\tilde{k}\left(\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{\mu^{\mathrm{f}}}+\frac{2}{K^{\mathrm{m}}}\right)\left(\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{\mu^{\mathrm{f}}}+\frac{2}{K^{\mathrm{f}}}\right)}. \tag{35}$$

Combining eqns (31)–(35), the effective shear modulus μ is

$$\frac{1}{\mu}=\frac{1}{\mu^{\mathrm{m}}}+\frac{2cA}{(1-cA)}\left(\frac{1}{\mu^{\mathrm{m}}}+\frac{1}{K^{\mathrm{m}}}\right). \tag{36}$$

The effective area bulk modulus can be obtained in a similar way by applying the remote stress $\sigma_{xx}^{0}=\sigma_{yy}^{0}$:

$$\frac{1}{K}=\frac{1}{K^{\mathrm{m}}}+c\left(\frac{1}{K^{\mathrm{f}}}-\frac{1}{K^{\mathrm{m}}}\right)\left[\frac{\frac{1}{K^{\mathrm{m}}}+\frac{1}{\mu^{\mathrm{m}}}}{\left(\frac{1}{K^{\mathrm{f}}}+\frac{1}{\mu^{\mathrm{m}}}\right)+c\left(\frac{1}{K^{\mathrm{m}}}-\frac{1}{K^{\mathrm{f}}}\right)}\right]. \tag{37}$$

Note that the expression for bulk modulus does not involve a sliding parameter k and is the same as for the perfect bonding case.

In contrast to the first two methods, the expressions for the elastic constants from the Mori–Tanaka method can be easily calculated by substituting directly the elastic constants of the matrix and inclusions into the final closed forms.

3.5. *Generalized self-consistent method*

The generalized self-consistent method (Mackenzie, 1950; Christensen and Lo, 1979), also called a three phase model, modifies the self-consistent model by placing the matrix phase between the inclusion and the effective medium. If we denote the radius of the inclusion by a and the radius of the concentric matrix core by b, we have a boundary value problem involving sliding boundary conditions (10)–(12) at $r = a$, perfect bonding conditions with continuity of tractions and displacements at $r = b$, and tractions $t_i = \sigma_{ij}^0 n_j$ applied at infinity.

The effective elastic moduli are obtained by equating the elastic strain energy stored in the heterogeneous medium with the elastic strain energy in the equivalent homogeneous medium (Christensen and Lo, 1979; Christensen, 1979)

$$W^0 = W^0 + \int_S (\sigma_{ij}^0 n_j u_i - \sigma_{ij} n_j u_i^0)\,\mathrm{d}S, \tag{38}$$

where the integral is taken over the surface $r = b$. When we evaluate the effective shear modulus, the boundary conditions involve σ_{rr}, $\sigma_{r\theta}$, u_r, u_θ and we have eight unknown constants. One of these constants vanishes as a result of eqn (38). This leads to a quadratic uncoupled equation for μ

$$C_1\left(\frac{\mu^{\mathrm{m}}}{\mu}\right)^2 + C_2\left(\frac{\mu^{\mathrm{m}}}{\mu}\right) + C_3 = 0. \tag{39}$$

The coefficients C_1, C_2, C_3 are given in Appendix A. These coefficients contain the sliding parameter k. It can be easily shown that, if k is taken to be infinite (perfect bonding), the solution of eqn (39) coincides with that of Christensen and Lo (1979).

The effective area bulk modulus is computed when applied loads are $\sigma_{xx}^0 = \sigma_{yy}^0$. The result is

$$\frac{1}{K}=\frac{1}{K^{\mathrm{m}}}+c\left(\frac{1}{K^{\mathrm{f}}}-\frac{1}{K^{\mathrm{m}}}\right)\left[\frac{\frac{1}{K^{\mathrm{m}}}+\frac{1}{\mu^{\mathrm{m}}}}{\frac{1}{K^{\mathrm{f}}}+\frac{1}{\mu^{\mathrm{m}}}+c\left(\frac{1}{K^{\mathrm{m}}}-\frac{1}{K^{\mathrm{f}}}\right)}\right], \tag{40}$$

which is exactly the same as the result of the Mori-Tanaka method, given in eqn (35). This also coincides with the result from the composite cylinder assemblage model.

The generalized self-consistent method was used earlier by Benveniste (1985) and Hashin (1990) to evaluate elastic constants of composites with spring-type interfaces.

4. THE CLM THEOREM

The Cherkaev–Lurie–Milton theorem (Cherkaev *et al.*, 1992), which we refer to as the CLM theorem, is a new result in 2D elasticity, that has very important applications in the mechanics of composite materials. It applies to linear elastic 2D materials with general anisotropy and arbitrary phase geometry.

For a two phase material, which is effectively isotropic and has homogeneous and isotropic components, it can be stated as follows. If the area bulk modulus K and the shear modulus μ of the components are transformed as

$$\frac{1}{K_t^m} = \frac{1}{K^m} + C, \quad \frac{1}{\mu_t^m} = \frac{1}{\mu^m} - C, \tag{41}$$

$$\frac{1}{K_t^f} = \frac{1}{K^f} + C, \quad \frac{1}{\mu_t^f} = \frac{1}{\mu^f} - C, \tag{42}$$

then

$$\frac{1}{K_t} = \frac{1}{K} + C, \quad \frac{1}{\mu_t} = \frac{1}{\mu} - C, \tag{43}$$

where the subscript t denotes the transformed moduli and C is a constant. Note that there is a restriction placed on C to ensure positive bulk and shear moduli K and μ. An important point is that under the transformation (41)–(42) the local stresses in an original and transformed material are the same under the same external tractions.

If we express the CLM theorem in terms of the area Young's modulus, we have

$$E_t = E, \tag{44}$$

which implies the invariance of E under the transformation.

One can show that all four effective medium theories studied in this paper do satisfy the CLM theorem for both perfect bonding and sliding. The original proof of the CLM theorem (Cherkaev *et al.* 1992) was based on the assumption of perfect bonding at the interfaces, but Moran and Gosz (1992) showed that the CLM transformation and theorem also holds for the constrained spring layer model in which sliding inclusion is a limit case. Following the main idea of their proof we include an alternate version of their proof in the Appendix B. The rigorous proof that the CLM transformation holds for pure sliding is given by Dundurs and Markenscoff (1993). These latest results prove an earlier conjecture by Thorpe and Jasiuk (1992) that the CLM theorem also holds for the more general boundary conditions such as sliding.

We discuss the consequences of the CLM theorem in the next section.

5. SPECIAL LIMIT CASES

Several limit cases are studied in this section to compare results of the effective medium theories and to observe the effects of sliding on the elastic properties of 2D composite materials.

5.1. *Materials with holes*

When the inclusions are holes, the effective moduli are easily obtained from the results of Section 3 by taking the limits $K^f \to 0$ and $\mu^f \to 0$. The results are independent of the value of interface parameter k, as expected.

We are particularly interested in the effective area Young's modulus and the area Poisson's ratio. These can be obtained from K and μ via eqns (5) and (6) as follows:

(1) Dilute result

$$\frac{E}{E^{\mathrm{m}}} = 1-3c, \quad \nu = \nu^{\mathrm{m}} - c(3\nu^{\mathrm{m}}-1). \tag{45}$$

(2) Self-consistent method

$$\frac{E}{E^{\mathrm{m}}} = 1-3c, \quad \nu = \nu^{\mathrm{m}} - c(3\nu^{\mathrm{m}}-1). \tag{46}$$

Note that eqns (46) and (45) are the same!

(3) Differential scheme

$$\frac{E}{E^{\mathrm{m}}} = (1-c)^3, \quad \nu = \tfrac{1}{3}+(\nu^{\mathrm{m}}-\tfrac{1}{3})(1-c)^3. \tag{47}$$

(4) Mori–Tanaka method

$$\frac{E}{E^{\mathrm{m}}} = \frac{1-c}{1+2c}, \quad \nu = \frac{c+\nu^{\mathrm{m}}(1-c)}{1+2c}. \tag{48}$$

(5) Generalized self-consistent method

$$\frac{E}{E^{\mathrm{m}}} = \frac{c(1-2c-c^2)+\sqrt{(1+c+c^2+c^3)^2-12c^2}}{(1+5c+2c^2)},$$

$$\nu = \left[\nu^{\mathrm{m}} - \frac{(1+c)}{(1-c)}\right]\frac{E}{E^{\mathrm{m}}}+1. \tag{49}$$

The expression for E for the generalized self-consistent method was taken from Christensen (1993).

A very interesting result has been recently obtained numerically by Day *et al.* (1992), namely that the effective area Young's modulus of a 2D material containing circular holes is *independent of* the Poisson's ratio of the host material. This observation is closely related to Michell's (1899) result, which states that the stress field in a material containing holes and subjected to tractions is *independent* of the elastic constants of the material, if the resultant of forces over every hole vanishes. This result can be easily proved by the CLM theorem (Cherkaev *et al.*, 1992; Day *et al.*, 1992). The CLM transformation leaves holes as holes and the change in Poisson's ratio of the matrix does not change E as seen from eqn (44). Since the CLM theorem holds for an arbitrary geometry, this result is true for holes of any shape. The application of the CLM theorem to the problem of a material with polygonal holes, including cracks, has been recently explored by Jasiuk *et al.* (1992b).

All the effective medium theories discussed in this paper predict that E is independent of ν as seen in eqns (45)–(49). We illustrate it in Fig. 1. Note that the predictions for E are different due to the different micromechanics models used but they all show the invariance of E on the Poisson's ratio of the matrix. The results from the numerical simulations show that the effect of geometric arrangement is significant as illustrated in Fig. 10 of Day *et al.* (1992) where E is given for regular honeycomb, regular triangular, and random overlapping inclusion geometries. We may add that the results for random nonoverlapping inclusions lie close to the random overlapping case (Thorpe, 1991). Comparing the results of Day for a random case with our Fig. 1 we see that the self-consistent method gives a closer prediction for E for small area fraction of holes than the other three methods. We should add that the simulations of Day are for single size holes while the effective medium theories may assume a gradation of sizes. However, since the theories discussed here are often used to predict

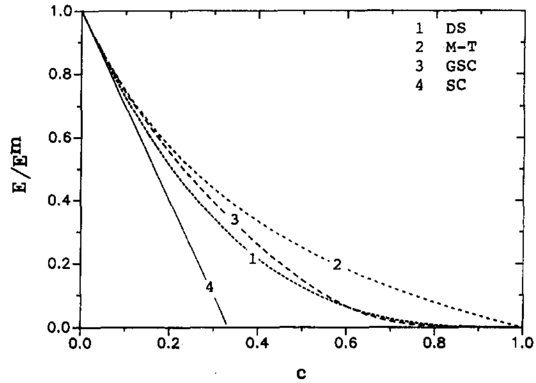

Fig. 1. Effective area Young's modulus E/E^m vs area fraction c of a material containing circular holes as predicted by four effective medium theories: self-consistent (SC), differential scheme (DS), Mori–Tanaka (M–T) and generalized self-consistent (GSC) methods. Note that the results are *independent* of Poisson's ratio of the matrix.

the effective elastic constants of composites with fibers of nearly equal diameters, this comparison is of interest.

Another interesting quantity is the effective 2D Poisson's ratio. A number of papers reported a tendency of Poisson's ratio to go to a fixed point as percolation is approached (i.e. $E \to 0$), and this result was recently proved by using the CLM theorem (Day *et al.*, 1992). When the material contains holes we can rewrite the CLM theorem in terms of E and ν as follows

$$\nu_t - \nu = (\nu_t^m - \nu^m)\frac{E}{E^m}. \tag{50}$$

Then if $E \to 0$, then the right-hand side vanishes and $\nu_t = \nu$, which shows that the effective area Poisson's ratio flows to a fixed point which is independent of the Poisson's ratio of the matrix. This point depends on the geometry (shape, relative size, and arrangement) of holes. We denote this point by ν_0.

Figure 2 illustrates the flow of area Poisson's ratio to the fixed point as the area fraction increases. The effective medium theories predict different values of ν_0. The differential scheme, the self-consistent method, and the Mori–Tanaka method predict that the effective Poisson's ratio ν goes to 1/3 as percolation point is approached (Fig. 2a), while the generalized self-consistent method $\nu_0 = 1$ (Fig. 2b). We define the volume (area) fraction at percolation by c_0. The percolation concentration for all the above methods is $c_0 = 1$ except for the self-consistent method which predicts $c_0 = \frac{1}{3}$. These can be easily obtained from eqns (46)–(49). c_0 is obtained by setting the expressions for (E/E^m) to zero, while ν_0 is evaluated by substituting c_0 into the expressions for ν. The results from the self-consistent method are shown in Fig. 3 of Thorpe and Jasiuk (1992).

Day *et al.* (1992) also looked at the flow of Poisson's ratio and found that the Poisson's ratio goes to the fixed point for all the geometric arrangements studied, and $\nu_0 = \frac{1}{3}$ for the triangular and random arrangements, and $\nu_0 = 1$ for the honeycomb network, as illustrated in Fig. 12 of Day *et al.* (1992). Therefore, the three effective medium theories predict the same ν_0 as the numerical simulations for the random case. The predictions for the area fraction at percolation c_0 differ as expected since the numerical simulations of Day *et al.* (1992) take inclusions of single size. However, the result for c_0 from the self-consistent method is much too low (Budiansky, 1965).

The existence of the fixed point ($\nu_0 = 0.2$) for Poisson's ratio for a 3D material with spherical holes was reported by Zimmerman (1991) for the differential scheme.

The effective elastic moduli of plates with holes near percolation were also discussed by Krajcinovic *et al.* (1992).

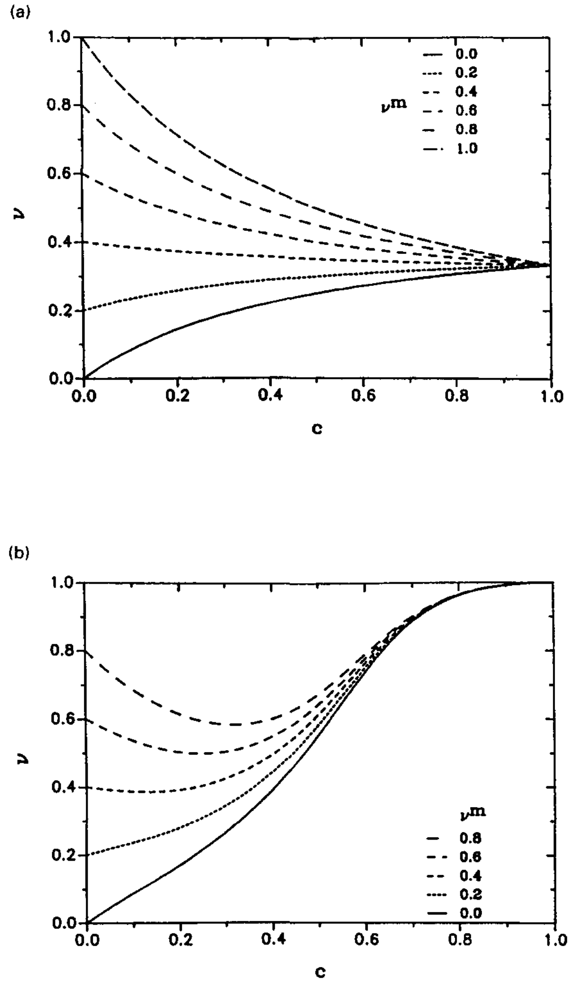

Fig. 2. Effective area Poisson's ratio ν vs c as predicted by (a) the Mori–Tanaka method and (b) the generalized self-consistent method.

5.2. *Materials with rigid inclusions*

In this section we consider materials with rigid inclusions. The results are computed by taking $K^{\mathrm{f}} \to \infty$ and $\mu^{\mathrm{f}} \to \infty$ from the expressions in Section 3 for the elastic inclusions. Here we supplement Jasiuk *et al.* (1992a) paper in which the results from the self-consistent method and the differential scheme were discussed. The final forms of the four theories are:

(1) Self-consistent method

$$\frac{1}{K} = \frac{1}{K^{\mathrm{m}}} - \frac{c}{K^{\mathrm{m}}}\left(1 + \frac{\mu}{K}\right), \tag{51}$$

$$\frac{1}{\mu} = \frac{1}{\mu^{\mathrm{m}}} + c\left(\frac{1}{\mu} + \frac{1}{K}\right)\left[\frac{\left(\frac{1}{\mu} - \frac{3}{\mu^{\mathrm{m}}}\right) - \frac{2}{\mu}\tilde{k}\frac{1}{\mu^{\mathrm{m}}}}{\left(\frac{2}{\mu} + \frac{3}{K}\right) + \frac{1}{\mu}\tilde{k}\left(\frac{1}{\mu} + \frac{2}{K}\right)}\right]. \tag{52}$$

(2) Differential scheme

$$\frac{\mu^{\mathrm{m}}}{\mu} = \left(\frac{(1-\nu^{\mathrm{m}})}{(1-\nu)}(1-c)^3\right)^{(1+\tilde{k})/(3+2\tilde{k})}, \tag{53}$$

$$\frac{K^{\mathrm{m}}}{K} = \frac{(1+\nu^{\mathrm{m}})(1-\nu)}{(1+\nu)(1-\nu^{\mathrm{m}})}\left(\frac{1-\nu^{\mathrm{m}}}{(1-\nu)}(1-c)^3\right)^{(1+\tilde{k})/(3+2\tilde{k})} \tag{54}$$

(3) Mori–Tanaka method

$$\frac{1}{K} = \frac{1}{K^{\mathrm{m}}} - \frac{c}{K^{\mathrm{m}}}\left[\frac{\dfrac{1}{\mu^{\mathrm{m}}}+\dfrac{1}{K^{\mathrm{m}}}}{\dfrac{1}{\mu^{\mathrm{m}}}+c\dfrac{1}{K^{\mathrm{m}}}}\right], \tag{55}$$

$$\frac{1}{\mu} = \frac{1}{\mu^{\mathrm{m}}} - 2A'\frac{c}{\mu^{\mathrm{m}}}\left[\frac{\dfrac{1}{\mu^{\mathrm{m}}}+\dfrac{1}{K^{\mathrm{m}}}}{1+A'\dfrac{c}{\mu^{\mathrm{m}}}}\right], \tag{56}$$

where

$$A' = \frac{1+\tilde{k}\dfrac{1}{\mu^{\mathrm{m}}}}{\dfrac{2}{\mu^{\mathrm{m}}}+\dfrac{3}{K^{\mathrm{m}}}+\tilde{k}\dfrac{1}{\mu^{\mathrm{m}}}\left(\dfrac{1}{\mu^{\mathrm{m}}}+\dfrac{2}{K^{\mathrm{m}}}\right)}. \tag{57}$$

(4) Generalized self-consistent method

$$\frac{1}{K} = \frac{1}{K^{\mathrm{m}}} - \frac{c}{K^{\mathrm{m}}}\left[\frac{\dfrac{1}{\mu^{\mathrm{m}}}+\dfrac{1}{K^{\mathrm{m}}}}{\dfrac{1}{\mu^{\mathrm{m}}}+c\dfrac{1}{K^{\mathrm{m}}}}\right], \tag{58}$$

$$C_1\left(\frac{\mu^{\mathrm{m}}}{\mu}\right)^2 + C_2\left(\frac{\mu^{\mathrm{m}}}{\mu}\right) + C_3 = 0, \tag{59}$$

where the coefficients C_1, C_2 and C_3 can be obtained from Appendix A.

The four effective medium theories predict very different results for the Poisson's ratio. In the perfect bonding limit, the self-consistent method and the differential scheme give the fixed value of Poisson's ratio, which is $\nu_0 = \frac{1}{3}$, as shown in Figs 2 and 3 in Jasiuk *et al.* (1992a). However, the Mori–Tanaka method and the generalized self-consistent method do not give the fixed value of Poisson's ratio as shown in Figs 3(a) and 4(a), but a tendency is there for the generalized self-consistent method. We should point out that it is not known whether there is a fixed point for the Poisson's ratio of the composite with rigid inclusions. Note that we cannot use the CLM theorem for this limiting case because it yields a trivial result.

As the interface parameter k decreases, i.e. sliding increases, the effective Poisson's ratio increases as predicted by all four methods. This observation was also made by Jasiuk *et al.* (1992a). For the limit case of pure sliding all theories show the fixed point of Poisson's ratio; this value is $\nu_0 = 1$ for all except the generalized self-consistent method which gives $\nu_0 = \frac{1}{2}$. The results for ν as predicted by the Mori–Tanaka theory and the three phase model

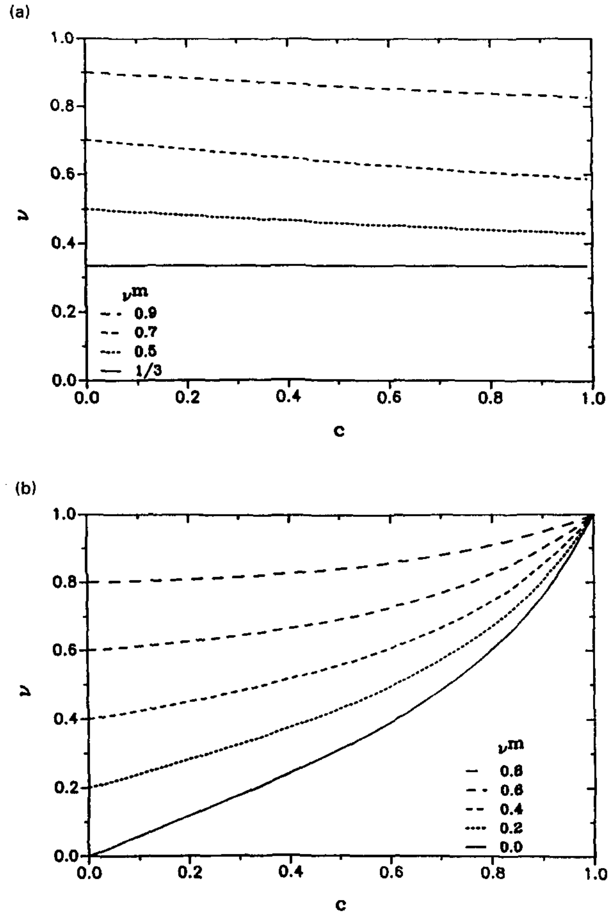

Fig. 3. Effective area Poisson's ratio ν vs c of a composite with rigid inclusions for (a) perfect bonding, and (b) sliding, given by the Mori–Tanaka method.

are shown in Figs 3–4. Unlike in the perfect bonding case, all theories, including the self-consistent method, show the percolation point at $c_0 = 1$. It is interesting to note that for the perfect bonding case the self-consistent method predicts percolation at $c_0 = \frac{2}{3}$, which agrees with the result for single size overlapping inclusions (Jasiuk *et al.*, 1992a).

When sliding takes place at the matrix-inclusion interface, the effective shear modulus and the effective area Young's modulus are lower than those of the perfect bonding case. All theories agree on this reduction of μ and E. However, the theories disagree on the area bulk modulus result. The self-consistent method and the differential scheme show a reduction of K while the Mori–Tanaka method and the generalized self-consistent method yield the same value of K as for the perfect bonding case. Another difference is found in the effective shear modulus. The Mori–Tanaka method predicts that μ is finite ($\mu = 3\mu^m$) at $c = 1$, while the other three methods give an infinite value of μ at $c = 1$.

When we compare the predictions from the effective medium theories with those from the numerical simulations (Thorpe, 1991) we find that for a material with perfectly bonded rigid inclusions the generalized self-consistent model agrees most closely with the numerical results for the shear modulus of a material with for nonoverlapping single-size inclusions, while the differential scheme is in closest agreement with the numerical results for the area bulk modulus of a material with the same geometry. The flow of Poisson's ratio is approximated well by the three phase model.

5.3. *Materials with equal shear moduli*

If the shear moduli of the matrix and the inclusions are the same, the upper and lower bounds coincide and the effective elastic constants are determined exactly (Hill, 1964;

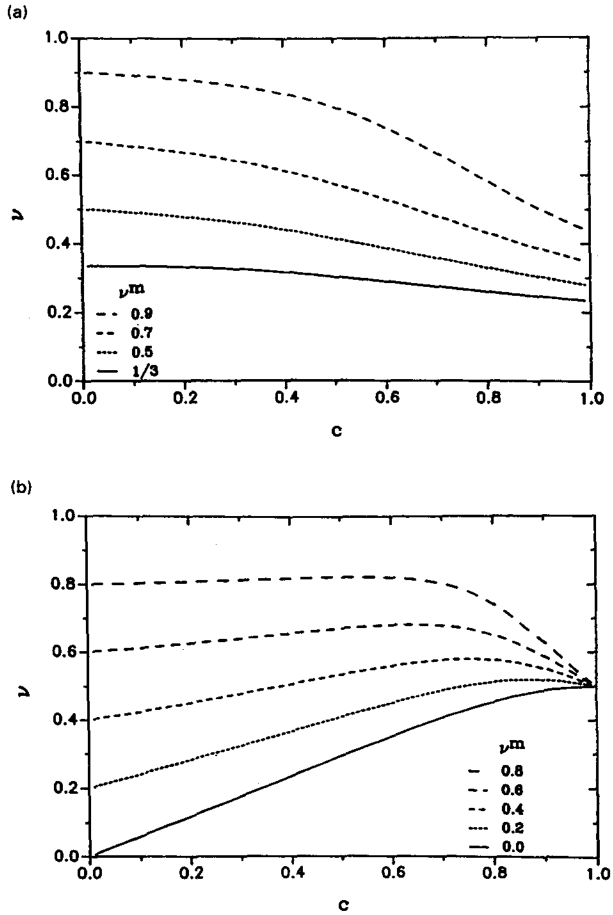

Fig. 4. Effective area Poisson's ratio ν vs c of a material with rigid inclusions for (a) perfect bonding, and (b) sliding, as predicted by the generalized self-consistent method.

Hashin, 1965). This is true in both 2D and 3D, but only for the perfect bonding case. All the effective medium theories used here give the exact values for elastic constants. The shear modulus is $\mu = \mu^{m} = \mu^{f}$ while the area bulk modulus is

$$K = K^{m} + c(K^{m} + \mu^{m})\left[\frac{K^{f} - K^{m}}{K^{f} + \mu^{m} + c(K^{m} - K^{f})}\right]. \tag{60}$$

It is interesting to note that these results, when expressed in terms of E and ν, obey the mixture law (Thorpe and Jasiuk, 1992)

$$E = cE^{f} + (1-c)E^{m}, \tag{61}$$

$$\nu = c\nu^{f} + (1-c)\nu^{m}. \tag{62}$$

These exact results can be proved for 2D by the CLM theorem as shown in Cherkaev *et al.* (1992) and Thorpe and Jasiuk (1992). All four effective medium theories reduce to (60)–(62) in this limiting case.

5.4. *Materials with equal Poisson's ratios*

If the Poisson's ratios of the matrix and the inclusions are the same, then the effective Poisson's ratio ν is in general different to that of the components. This problem was also investigated by Snyder *et al.* (1992) who reported the results from the self-consistent method

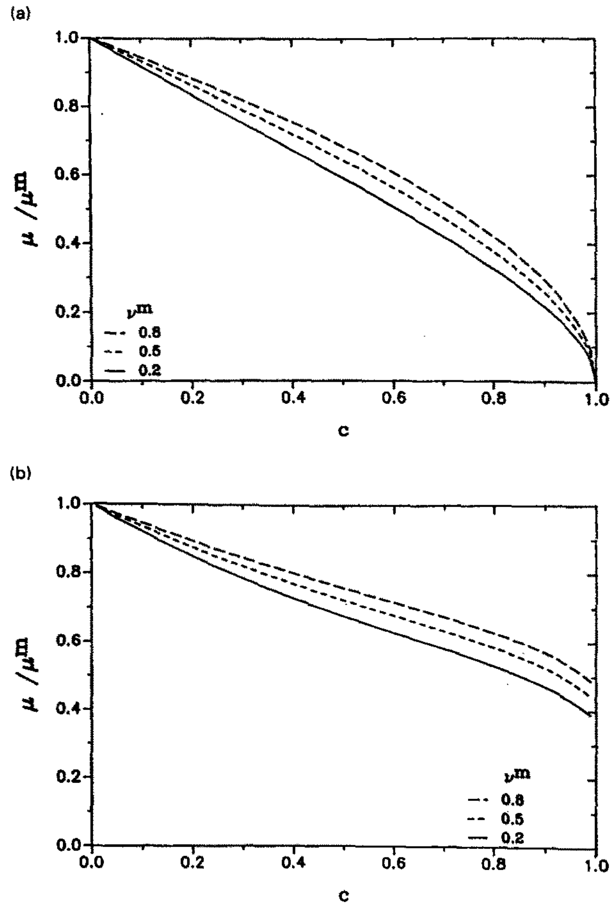

Fig. 5. Effective shear modulus μ/μ^{m} vs c for a homogeneous material with sliding inclusions as predicted by (a) the differential scheme, and (b) the generalized self-consistent method.

and the numerical simulations. It is interesting to note that for the perfect bonding case with $E^{m} \neq E^{f}$ both the numerical simulations and the predictions of three effective medium theories give a result that if $\nu^{m} = \nu^{f} > \frac{1}{3}$, then ν is less than ν^{m} and ν^{f}, if $\nu^{m} = \nu^{f} = \frac{1}{3}$, then $\nu = \frac{1}{3}$ and if $\nu^{m} = \nu^{f} < \frac{1}{3}$, then ν is greater than $\nu^{m} = \nu^{f}$, as illustrated in Fig. 6(a) for the Mori–Tanaka method. The generalized self-consistent method deviates a little from this rule [Fig. 6(b)]. When the mismatch in E is large this effect is more pronounced, while for a homogeneous material it disappears.

5.5. *Materials with sliding inclusions having same elastic constants as the matrix*

It is known that the stress field is *independent* of the elastic constants when the matrix and sliding inclusions are made of the same material and the loading is in terms of prescribed tractions (Dundurs and Stippes, 1970). Recently, Thorpe and Jasiuk (1992) showed that the area Young's modulus E of such a material is *independent* of the Poisson's ratio for the dilute concentration of inclusions. Here, we find that the four effective medium theories studied in this paper also predict this result as shown below:

(1) Dilute result

$$\frac{E}{E^{m}} = 1-\frac{c}{2}, \quad \nu = \nu^{m}+\frac{c}{2}(1-\nu^{m}). \tag{63}$$

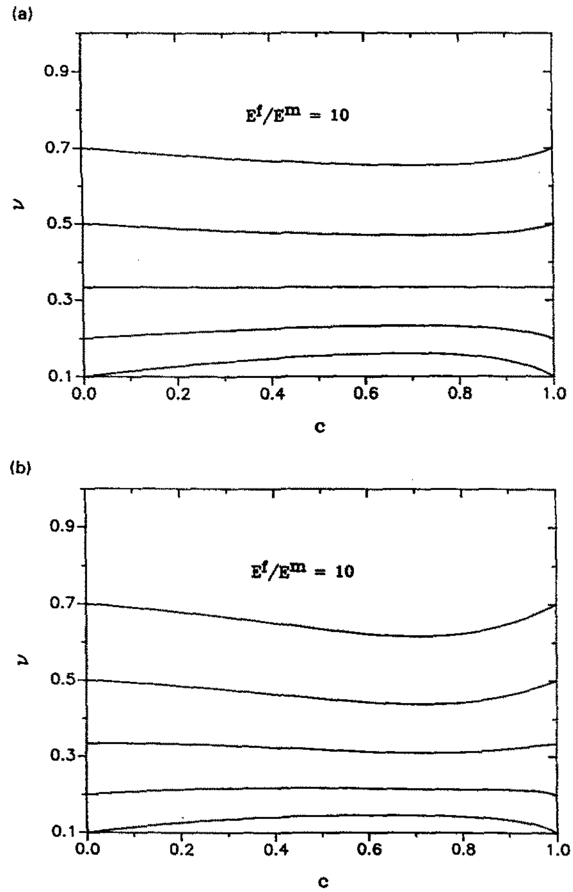

Fig. 6. Effective area Poisson's ratio ν vs c when $E^f/E^m = 10$ as predicted by (a) the Mori–Tanaka method, and (b) the generalized self-consistent method.

(2) Self-consistent method

$$\frac{E}{E^m} = 1-\frac{c}{2}, \quad \nu = 1-(1-\nu^m)\left(1-\frac{c}{2}\right). \tag{64}$$

(3) Differential scheme

$$\frac{E}{E^m} = (1-c)^{1/2}, \quad \nu = 1-(1-\nu^m)(1-c)^{1/2}. \tag{65}$$

(4) Mori–Tanaka method

$$\frac{E}{E^m} = \frac{4-c}{4+c}, \quad \nu = 1-(1-\nu^m)\left(\frac{4-c}{4+c}\right). \tag{66}$$

(5) Generalized self-consistent method

$$\frac{E}{E^m} = \frac{c[3(1-2c)^2-1]+2\sqrt{4-3(1-2c)^2c^2}}{c[3(1-2c)^2+1]+4}, \quad \nu = \nu^m\frac{E}{E^m}. \tag{67}$$

We illustrate the results for E as predicted by the four methods in Fig. 7. The invariance of E on ν can be proved by the CLM theorem in the same way as the proof for material with holes was done. We may add that eqn (50) holds for this case also. The effective area Poisson's ratios predicted by the effective medium theories do not go to a fixed point, except

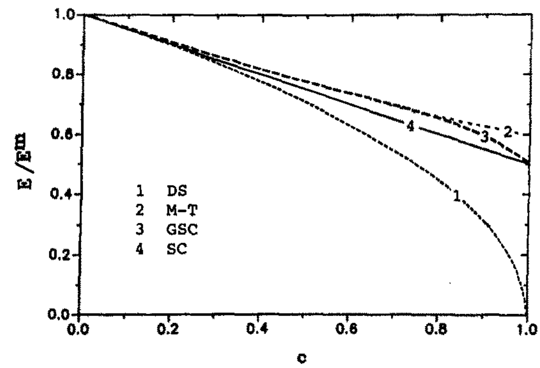

Fig. 7. Effective area Young's modulus E/E^m vs c for a material with sliding inclusions and $E^m = E^f$ and $\nu^m = \nu^f$ as predicted by the effective medium theories. Note that the result is *independent* of Poisson's ratio of the host material.

for the differential scheme result, which gives $\nu_0 = 1$. Here, we do not expect a fixed point unless E goes to zero [see eqn (50)]. The effective shear modulus decreases considerably due to sliding, but this reduction is most pronounced in the differential scheme result which predicts that $\mu \to 0$ at $c = 1$. The interpretation of this result is difficult but one can imagine that at $c = 1$ the matrix disappears and is replaced by a network of vanishingly thin sliding interfaces which do not resist shear.

The effective elastic moduli of a material with sliding inclusions of the same material as the matrix in 2D and 3D were evaluated earlier by Shibata *et al.* (1990) by using the Mori–Tanaka method.

5.6. *Material with elastic inclusions*

Finally, we consider a general case of elastic inclusions and we focus on a physical problem of a glass/epoxy composite with $K^f/K^m = 11$ and $\mu^f/\mu^m = 22$. We choose this system because we have available the numerical results of Chen *et al.* (1993), and second order bounds (Hill, 1964; Hashin, 1965), and third order bounds (Torquato and Lado, 1988) for the perfect bonding case. We illustrate the predictions from the effective medium theories for K and μ for both perfect bonding (Fig. 8) and pure sliding (Fig. 9). Note a significant difference in the predictions for μ for perfect bonding and sliding cases. The results of numerical simulations for perfect bonding case, given in Fig. 4 of Chen *et al.* (1993), lie between the predictions of the self-consistent method and the differential scheme. It may also be pointed out that the predictions of the Mori–Tanaka method lie outside Torquato's third order bounds.

6. CONCLUSIONS

The effective moduli of two dimensional composites with sliding interfaces are calculated using four effective medium theories: the self consistent method, the differential scheme, the Mori–Tanaka method, and the generalized self-consistent method. We also discuss the results for composites with perfectly bonded inclusions and for materials with holes, for completeness.

We show that the effect of sliding is significant but we point out that the actual predictions may vary considerably depending on the method used.

All four methods predict a reduction in the effective area Young's modulus and shear modulus due to sliding. However, only the self-consistent method and the differential scheme give the reduction in the effective area bulk modulus. All methods show an increase in the Poisson's ratio due to sliding.

All theories give a result that the 2D Young's modulus of a material with holes is independent of the Poisson's ratio of the host material. The same conclusion holds for a material with sliding inclusions having the same elastic constants as the matrix. These agree

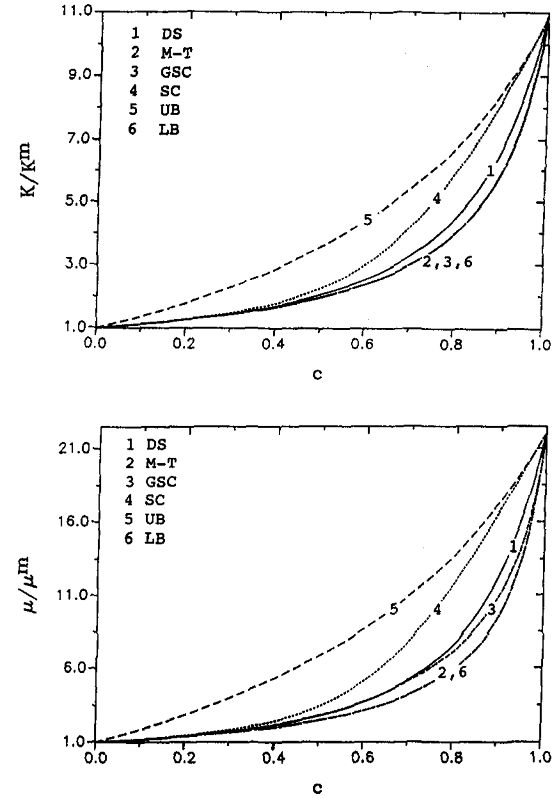

Fig. 8. Effective shear modulus μ/μ^m and area bulk modulus K/K^m vs c for a glass-epoxy composite for perfect bonding case, as predicted by the four effective medium theories and second order upper (UB) and lower (LB) bounds.

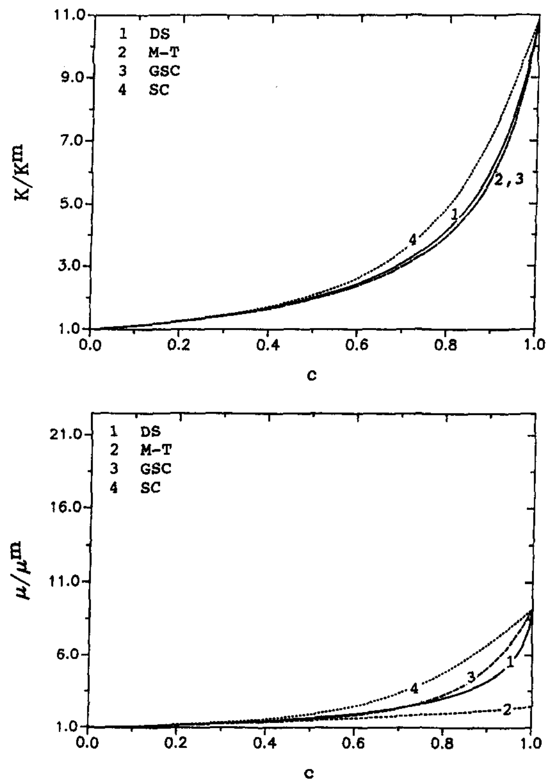

Fig. 9. Effective shear modulus μ/μ^m and area bulk modulus K/K^m vs c for glass-epoxy composite for pure sliding case, as predicted by the four effective medium theories.

with the results from the CLM theorem. However, the actual values given by each effective medium theory differ as they employ different geometric models.

When the shear moduli of the components are equal the exact result is known. All theories reduce to this result.

Also, we compare the results from the effective medium theories with those from the numerical simulations for composites with randomly distributed nonoverlapping single-size circular inclusions with perfectly bonded interfaces. We find that no one method gives close predictions for whole parameter space, but they yield close approximations for a given range of parameters.

In our discussions we considered the whole range of inclusion area fractions, including $c = 1$. This is justified as the effective medium theories may allow gradation of sizes. Also, this limit is of interest, as the predictions of the methods differ most in this limit.

Finally, all four theories do obey the CLM theorem for both the perfect bonding and sliding cases.

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APPENDIX A

The coefficients of the quadratic equation for the shear modulus given by the generalized self-consistent method [eqn (39)] are

$$C_1 = c_1 e_1 - b_1 f_1, \tag{A1}$$

$$C_2 = c_1 e_2 + c_2 e_1 - (b_1 f_2 + b_2 f_1), \tag{A2}$$

$$C_3 = c_2 e_2 - b_2 f_2, \tag{A3}$$

where

$$b_1 = x_1 + x_2, \tag{A4}$$

$$b_2 = -x_1 + \eta_m x_2, \tag{A5}$$

$$c_1 = -x_3 + 2x_2 - x_4, \tag{A6}$$

$$c_2 = -\eta_m x_3 - 2x_2 + x_4, \tag{A7}$$

$$e_1 = x_5 - 3x_6 - x_7, \tag{A8}$$

$$e_2 = -x_5 + 3x_6 - \eta_m x_7, \tag{A9}$$

$$f_1 = 2x_6 - 2x_7 + x_8, \tag{A10}$$

$$f_2 = 2\eta_m x_6 + 2x_7 - x_8, \tag{A11}$$

where

$$x_1 = [(\eta^f - 3)\mu^m - (\eta^m - 3)\mu^f]c^3 - 3(\mu^f - \mu^m)c^2, \tag{A12}$$

$$x_2 = \mu^f + \eta^f \mu^m, \tag{A13}$$

$$x_3 = 2(\mu^f - \mu^m)c^2, \tag{A14}$$

$$x_4 = 2[(\eta^f + 1)\mu^m + (\eta^m + 1)\mu^f]c, \tag{A15}$$

$$x_5 = \{6\mu^m\mu^f - \tilde{k}[(\eta^m + 3)\mu^f - (\eta^f + 3)\mu^m]\}c^3, \tag{A16}$$

$$x_6 = [\mu^m\mu^f - \tilde{k}(\mu^f - \mu^m)]c^2, \tag{A17}$$

$$x_7 = 3\mu^m\mu^f + \tilde{k}(\mu^f + \eta^f\mu^m), \tag{A18}$$

$$x_8 = 2\{2\mu^m\mu^f - \tilde{k}[(\eta^m - 1)\mu^f - (\eta^f - 1)\mu^m]\}c, \tag{A19}$$

where c is the volume fraction of inclusions and $\eta^{f,m} = 1 + 2(\mu^{f,m})/(K^{f,m})$.

APPENDIX B

Consider a two-phase composite, with sliding boundary conditions at interfaces (10)–(12), subjected to uniform surface tractions

$$t_i = \sigma_{ij}^0 n_j, \tag{B1}$$

where σ_{ij}^0 is a constant tensor. Then, the volume average of stress, $\langle\sigma_{ij}\rangle$, is

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\int_D \sigma_{ij}\,\mathrm{d}V = \frac{1}{V}\left(\int_{D-\Omega}\sigma_{ij}\,\mathrm{d}V + \int_\Omega \sigma_{ij}\,\mathrm{d}V\right) = \sigma_{ij}^0 \tag{B2}$$

and the effective compliance of the composite S_{ijkl} is defined by

$$\langle\varepsilon_{ij}\rangle = S_{ijkl}\sigma_{kl}^0, \tag{B3}$$

where $\langle\varepsilon_{ij}\rangle$ is the volume average strain, which for the two phase composite with sliding interfaces is defined by

$$\langle\varepsilon_{ij}\rangle = \frac{1}{V}\left\{\int_{D-\Omega}\varepsilon_{ij}\,\mathrm{d}V + \int_\Omega \varepsilon_{ij}\,\mathrm{d}V + J_{ij}\right\}, \tag{B4}$$

where

$$J_{ij} = \frac{1}{2}\int_{|\Omega|}([u_i]n_j + [u_j]n_i)\,\mathrm{d}S \tag{B5}$$

and $D-\Omega$ is the matrix domain, Ω denotes the inclusions' regions and $|\Omega|$ implies the surfaces of inclusions. The stresses and strains in the individual phases are related by the Hooke's law as

$$\varepsilon_{ij} = S_{ijkl}^m \sigma_{kl} \quad \text{in } D-\Omega, \tag{B6}$$

$$\varepsilon_{ij} = S_{ijkl}^f \sigma_{kl} \quad \text{in } \Omega. \tag{B7}$$

Substituting eqns (B6)–(B7) into (B4) and (B3) we have

$$\frac{1}{V}\left[S_{ijkl}^m \int_{D-\Omega}\sigma_{kl}\,\mathrm{d}V + S_{ijkl}^f \int_\Omega \sigma_{kl}\,\mathrm{d}V + J_{ij}\right] = S_{ijkl}\sigma_{kl}^0. \tag{B8}$$

Next, consider a transformed system, denoted by the subscript t, such that the stress remains invariant

$\sigma_{\gamma}^{\mathrm{t}}(\mathbf{x})\sigma_{ij}^{\mathrm{t}}(\mathbf{x}) = \sigma_{ij}(\mathbf{x})$. The compliances of such a system are

$$S_{ijkl}^{\mathrm{mt}} = S_{ijkl}^{\mathrm{m}} + D_{ijkl}, \tag{B9}$$

$$S_{ijkl}^{\mathrm{ft}} = S_{ijkl}^{\mathrm{f}} + D_{ijkl}, \tag{B10}$$

where

$$D_{ijkl} = \frac{C}{2}\left[\frac{1}{2}\delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})\right], \tag{B11}$$

as given by Cherkaev *et al.* (1992). We refer to this step as the CLM transformation. The proof that the CLM transformation also holds for the composites with sliding interfaces is given by Moran and Gosz (1992) and Dundurs and Markenscoff (1993). For the transformed system eqn (B8) becomes

$$\frac{1}{V}\left[S_{ijkl}^{\mathrm{mt}}\int_{D-\Omega}\sigma_{kl}\,\mathrm{d}V + S_{ijkl}^{\mathrm{ft}}\int_{\Omega}\sigma_{kl}\,\mathrm{d}V + J_{ij}^{\mathrm{t}}\right] = S_{ijkl}^{\mathrm{t}}\sigma_{kl}^{0}. \tag{B12}$$

Subtracting eqn (B8) from (B12) we have

$$\frac{1}{V}\left[(S_{ijkl}^{\mathrm{mt}} - S_{ijkl}^{\mathrm{m}})\int_{D-\Omega}\sigma_{kl}\,\mathrm{d}V + (S_{ijkl}^{\mathrm{ft}} - S_{ijkl}^{\mathrm{f}})\int_{\Omega}\sigma_{kl}\,\mathrm{d}V\right] = (S_{ijkl}^{\mathrm{t}} - S_{ijkl})\sigma_{kl}^{0}, \tag{B13}$$

which can be contracted to

$$\frac{1}{V}\left[D_{ijkl}\left(\int_{D-\Omega}\sigma_{kl}\,\mathrm{d}V + \int_{\Omega}\sigma_{kl}\,\mathrm{d}V\right)\right] = (S_{ijkl}^{\mathrm{t}} - S_{ijkl})\sigma_{kl}^{0} \tag{B14}$$

by using eqns (B9)–(B10). Note that the quantity $J_{ij}^{\mathrm{t}} - J_{ij} = 0$. Since the stress fields are the same for both systems and the spring constant k remains unchanged, then the jumps in displacements also remain unchanged due to eqn (12) (Moran and Gosz, 1992). Finally using eqn (B14) and the definition of the average stress (B2) we have

$$S_{ijkl}^{\mathrm{t}} = S_{ijkl} + D_{ijkl}. \tag{B15}$$